System: A system is an arrangement or combination of different physical components such that it gives the proper output to given input. A kite is an example of a physical system, because it is made up of paper and sticks. A classroom is an example of a physical system.

Control: The meaning of control is to regulate, direct or command a system so that a desired objective is obtained.

Plant: It is defined as the portion of a system which is to be controlled or regulated. It is also called a process

Controller: It is the element of the system itself, or may be external to the system. It controls the plant or the process.

Input: The applied signal or excitation signal that is applied to a control system to get a specified output is called input.

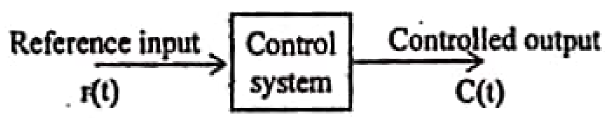
Output: The actual response that is obtained from a control system due to the application of the input is termed as output.

Disturbances: The signal that has some adverse effect on the value of the output of a system is called disturbance. If a disturbance is produced within the system, it is termed as an internal disturbance: otherwise, it is known as an external disturbance.

Control Systems: It is an arrangement of different physical components such that it give the desire output for the given input by means of regulate or control either direct or indirect method.

A control system must have (1) input, (2) output, (3) ways to achieve input and output objectives and (4) control action.

Fig. The following shows the cause-and-effect relationship between the input and the output.



Any system can be characterized mathematically by (1) Transfer function (2) State model

$$\text{Transfer Function} = \left.\frac{\text{Laplace transform of output}}{\text{Laplace transform of input}}\right|_{\text{initial conditions} = 0}$$

$$= \frac{L[c(t)]}{L[r(t)]} = \left.\frac{C(s)}{R(s)}\right|_{\text{initial conditions} = 0}$$

Transfer function is also called impulse response of the system.

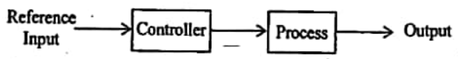
$$C(s) = T.F. \times R(s)$$

Classification of Control System :

- Open-loop C.S
- Closed-loop (or) feedback C.S
  1) Positive feedback C.S
  2) Negative feedback C.S

Open-loop Control System :

The Open-loop control system can be described by a block diagram as shown in the figure.



The reference input controls the output through a control action process. In the block diagram shown, it is observed the output has no effect on the control action. Such a system is termed as open-loop control system.

In an open-loop control system, the output is neither measured nor fed-back for comparison with the input. Faithfulness of an open-loop control system depends on the accuracy of input calibration.

Examples for open – loop control systems are traffic lights, fans, any system which is not having the sensor.

Advantages and disadvantages of open – loop system:

Advantages

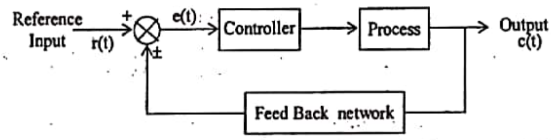
- These systems are simple in construction and design.
- These systems are economic.
- These systems are easy from maintenance point of view.
- Usually these systems are not much troubled with problems of stability.
- These systems are convenient to use when output is difficult to measure.

Disadvantages

- These systems are not accurate and reliable because their accuracy is dependent on the accuracy of calibration.
- In these systems, inaccurate results are obtained with parameter variations, i.e., internal disturbances.
- Recalibration of the controller is required from time to time for maintaining quality and accuracy

Closed-loop Control System :

In a closed-loop control system, the output has an effect on control action through a feedback as shown and hence closed-loop control systems are also termed as feedback control systems. The control action is actuated by an error signal 'e(t)' which is the difference between the input signal 'r(t)' and the output signal 'c(t)'. This process of comparison between the output and input maintains the output at a desired level through control action process.



The control systems without involving human intervention for normal operation are called automatic control systems. A closed-loop (feedback) control system using a power amplifying device prior to controller and the output of such a system being mechanical i.e., position, velocity, acceleration is called servomechanism.

Advantages and disadvantages of closed – loop system

Advantages

- In these systems accuracy is very high due to correction of any arising error.
- Since these systems sense environmental changes as well as internal disturbances, the errors are modified.
- There is reduced effect of non – linearity in these systems.
- These systems have high bandwidth, i.e., high operating frequency zone.
- There are facilities of automation in these systems.

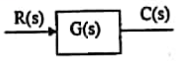
Disadvantages

- These systems are complicated in design and, hence, costlier.
- These systems may be unstable.

Comparison of Open-loop and Closed-loop Control systems :

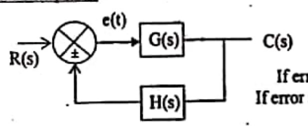
Open – loop C.S.	Closed – loop C.S.
1. The accuracy of an open-loop system depends on the calibration of the input. Any departure from pre – determined calibration affects the output.	1. As the error between the reference input and the output is continuously measured through feedback, the closed – loop system works more accurately.
2. The open – loop system is simple to construct and cheap	2. The closed – loop system is complicated to construct and costly
3. The open – loop systems are generally stable.	3. The closed – loop systems can become unstable under certain conditions
4. The operation of open – loop system is affected due to presence of non linearity's in its elements.	4. In terms of the performance, the closed – loop systems adjusts to the effects of non – linearity's present in its elements.

Open-loop C.S. :

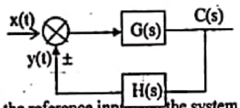


$$\frac{C(s)}{R(s)} = G(s) \quad \text{or} \quad C(s) = G(s)\,R(s)$$

Closed-loop C.S. :



If error signal e(t) is zero, output is controlled.
If error signal is not zero, output is not controlled.



For Positive feedback, error signal $= x(t) + y(t)$

For Negative feedback, error signal $= x(t) - y(t)$

The purpose of feedback is to reduce the error between the reference input and the system output.

+Ve feedback: Unity F/B (H(s) = 1) | Non unity F/B(H(s) ≠ 1)

-Ve feedback: Unity F/B | Non unity F/B

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)}; \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s)\,H(s)}; \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}; \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)\,H(s)}$$

Where $G(s)$ = T.F without feedback (or) T.F of the forward path

$H(s)$ = T.F of the feedback path

The feedback has effects on such system performance characteristics as stability, bandwidth, overall gain, impedance and sensitivity.

1.1 Effect of feedback on Stability :

Stability is a notion that describes whether the system will be able to follow the input command. A system is said to be unstable, if its output is out of control or increases without bound. It can be demonstrated that one of the advantages of incorporating feedback is that it can stabilize an unstable system.

Effect of feedback on Overall gain :

Feedback affects the gain G of a non-feedback system by a factor of $1 \pm GH$. The general effect of feedback is that it may increase or decrease the gain. In a practical control system, G and H are functions of frequency, so the magnitude of $1 + GH$ may be greater than 1 in one frequency range but less than 1 in another. Therefore, feedback could increase the gain of the system in one frequency range but decrease it in another.

1.2 Effect of feedback on Sensitivity:

Consider G as a parameter that may vary. The sensitivity of the gain of the overall system M to the variation in G is defined as

$$S_G^M = \frac{\partial M / M}{\partial G / G}$$

where ∂M denotes the incremental change in M due to the incremental change in G; $\partial M / M$ and $\partial G / G$ denote the percentage change in M and G, respectively.

$$S_G^M = \frac{\partial M}{\partial G} \cdot \frac{G}{M} = \frac{1}{1 + GH}$$

This relation shows that the sensitivity function can be made arbitrarily small by increasing GH, provided that the system remains stable. In an open-loop system, the gain of the system will respond in a one-to-one fashion to the variation in G. In general, the sensitivity of the system gain of a feedback system to parameter variations depends on where the parameter is located.

The effects of feedback are as follows.

(i) Gain is reduced by a factor
(ii) There is reduction of parameter variation by a factor $1 + G(s)H(s)$.
(iii) There is improvement in sensitivity.
(iv) There may be reduction of stability.

The disadvantages of reduction of gain and reduction of stability can be overcome by gain amplification and good design, respectively.

For a complicated system, it is easy to find the transfer function of each and every element, and output of a certain block may act as an input to other block or blocks. Therefore, the knowledge of transfer function of each block is not sufficient in this case. The interrelation between the elements is required to find the overall transfer function of the system. There are two method : (1) Block diagram and (2) signal flow graph.

CHAPTER - 2 BLOCK DIAGRAM S AND SIGNAL FLOW GRAPHS

There are two methods: (1) by using Block diagram or (2) Signal flow graph, to find the overall transfer function of a big complicated control system.

2.1 BLOCK DIAGRAM ALGEBRA

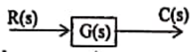
Block diagram reduction techniques:

Some of the important rules for block diagram reduction techniques are given below :

1. The block diagram shown below relates the output and input as per the transfer function relation given below :

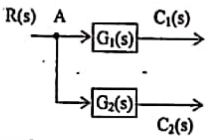
$$G(s) = \frac{C(s)}{R(s)} \qquad \text{(or)} \qquad C(s) = R(s) \cdot G(s)$$

where G(s) is known as the transfer function of the system.



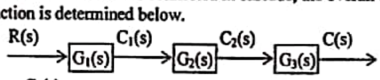
2. **Take off point :**

Application of one input source to two or more systems is represented by a take off point as shown at point A in the below figure.



3. **Blocks in cascade :**

When several blocks are connected in cascade, the overall equivalent transfer function is determined below.



$$\frac{C_1(s)}{R(s)} = G1(s)$$

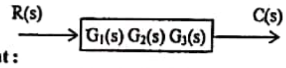
$$\frac{C_2(s)}{C_1(s)} = G2(s)$$

$$\frac{C(s)}{C_2(s)} = G3(s)$$

Multiplying above three equations, the equivalent transfer function is

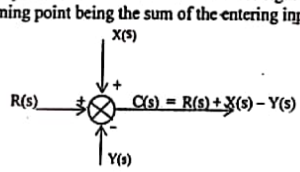
$$\frac{C(s)}{R(s)} = G_1(s)\, G2(s)\, G3(s)$$

The equivalent diagram is given by

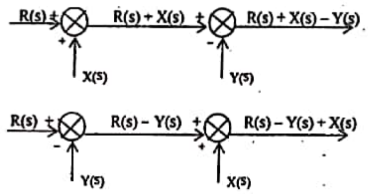


4. **Summing point :**

Summing point represents summation of two or more signal entering in a system. The output of a summing point being the sum of the entering inputs.

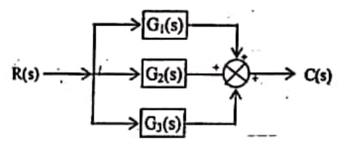


5. **Interchanging summing points:** Consecutive summing points can be interchanged, as this interchange does not alter the output signal.



6. **Blocks in parallel:**

When one or more blocks are connected in parallel, the overall equivalent transfer function is determined below.



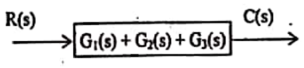
$$C(s) = R(s)\, G_1(s) + R(s)\, G_2(s) + R(s)\, G_3(s)$$

$$\text{or} \quad C(s) = R(s)\left\{G_1(s) + G_2(s) + G_3(s)\right\}$$

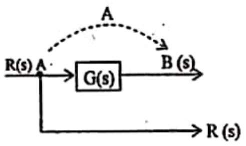
Therefore, the overall equivalent transfer function is,

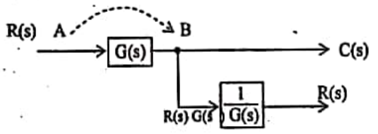
$$\frac{C(s)}{R(s)} = [\, G_1(s) + G_2(s) + G_3(s) \,]$$

The equivalence of above diagram is

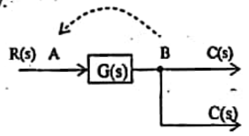


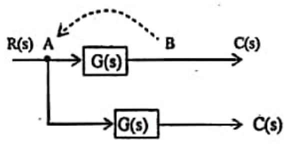
7. Shifting of a take off point from a position before a block to a position after the block is shown below.



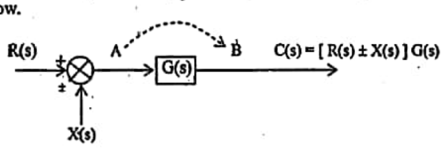


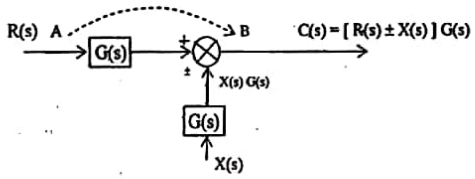
8. Shifting of a take off point from a position after a block to a position before the block is shown below.



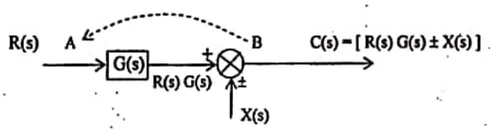


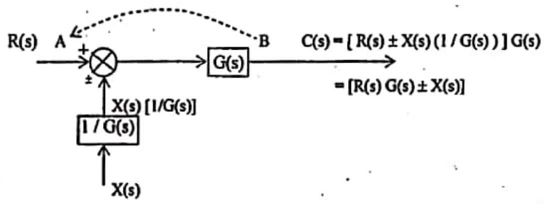
9. Shifting of a summing point from a position before a block to a position after the block is shown below.



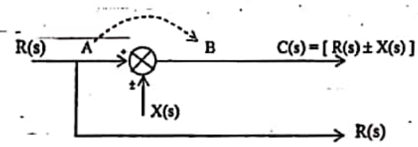


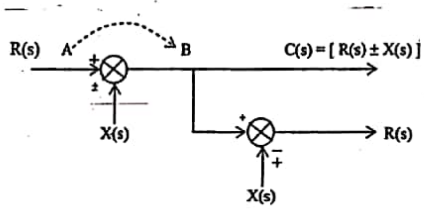
10. Shifting of a summing point from B position after a block to a position A before the block is shown below.



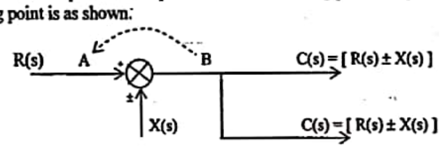


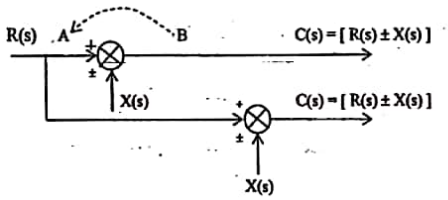
11. Shifting of a take off point from A position before a summing point to a position B after the summing point is shown below.





12. Shifting a take off point from a position after a summing point to a position before the summing point is as shown:





2.2 SIGNAL FLOW GRAPHS

A signal flow graph may be defined as a graphical means of portraying the input-output relationships between the variables of a set of linear algebraic equations.

Consider that a linear system is described by the set of N algebraic equations

$$y_j = \sum_{k=1}^{N} a_{kj} y_k \qquad j = 1, 2, \ldots\ldots, N$$

Basic properties of signal flow graphs :

1. A signal flow graphs applies only to linear systems.
2. The equations based on which a signal flow graph is drawn must be algebraic equations in the form of effects as functions of causes.
3. Nodes are used o represent variable. Normally, the nodes are arranged from left to right, following a success of causes and effects through the system.
4. Signals travel along branches only in the direction described by the arrows of the branches.
5. The branch directing from node y_k to y_j represents the dependence of the variable y_k upon y_j, but not the reverse.
6. A signal Y_k traveling along a branch between nodes y_k and y_j is multiplied by the gain of the branch, a_{kj}, so that a signal $a_{kj}\, y_k$ is delivered at node y_j.

Definitions for Signal Flow Graphs:

Input Node (Source): An input node is a node that has only outgoing branches.

Output Node (Sink): An output node is a node which has only incoming branches.

Path: A path is any collection of a continuous succession of branches traversed in the same direction.

Forward Path: A forward path is a path that starts at an input node and ends at an output node and along which no node is traversed more than once.

Loop: A loop is a path that originates and terminates on the same node and along which no other node is encountered more than once.

Path gain: The product of the branch gains encountered in traversing a path is called the path gain.

Forward path gain: Forward path gain is defined as the path gain of a forward path.

Loop gain: Loop gain is defined as the path gain of a loop.

Masons Gain formula:

The general gain formula is

$$M = \frac{Y_{out}}{Y_{in}} = \sum_{k=1}^{N} \frac{M_k \Delta_k}{\Delta}$$

where

M = gain between y_{in} and y_{out}

y_{out} = output node variable

y_{in} = input node variable

N = total number of forward paths

M_k = gain of the k^{th} forward path

$$\Delta = 1 - \sum_m P_{m1} + \sum P_{m2} - \sum P_{m3} + \ldots\ldots$$

= 1 − (sum of all individual loop gains) + (sum of gain products of all possible combinations of two non-touching loops) − (sum of the gain products of all possible combinations of three non-touching loops) +......

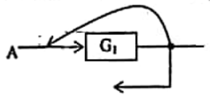
P_{mr} = gain product of the m^{th} possible combination of 'r' non-touching loops

Δ_k = the Δ for the part of the signal flow graph which is non-touching with the k^{th} forward path

Objective Questions

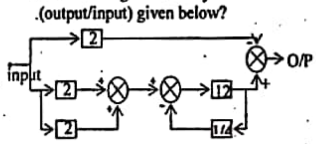
01. The block diagram contains
(a) system output variable
(b) system input variable
(c) the functional relations of the variables
(d) all the above

02. In a block diagram, when a take off point is moved ahead of a block, G_1.



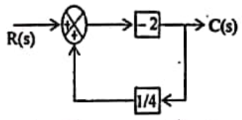
(a) The block G_1 will be added in parallel.
(b) The block G_1 will be added in the forward path.
(c) The block G_1 will be added in series.
(d) The block G_1 will be added in the feedback path.

03. What is the gain of the system (output/input) given below?



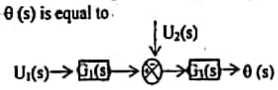
(a) 36 (b) 10
(c) 90 (d) –10

04. The closed-loop gain of the system sketched below is



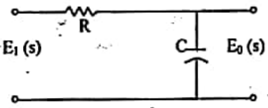
(a) –4/3 (b) –4
(c) 4 (d) 4/3

05. In the block diagram shown, the output θ (s) is equal to



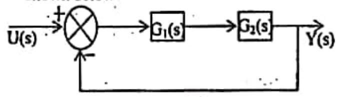
(a) $U_1(s) + U_2(s)$
(b) $U_1(s)\,G(s)$
(c) $G_1(s)\,G_3(s)\,U_1(s) - G_3(s)\,U_2(s)$

06. The transfer function $E_0(s)/E_1(s)$ of the RC-network shown is given by



(a) $\frac{1}{RCS+1}$ (b) $\frac{1}{RCS}$
(c) $\frac{RCS}{RCS+1}$ (d) None

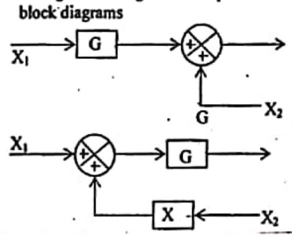
07. The block diagram of a certain system is shown below



The transfer function Y(s)/U(s) is given by

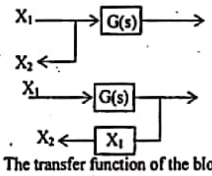
(a) $\frac{G_1(s)\,G_2(s)}{1 - G_1(s)\,G_2(s)}$
(b) $G_1(s)\,G_2(s)$
(c) $\frac{1 + G_1(s)\,G_2(s)}{G_1(s)\,G_2(s)}$
(d) $\frac{G_1(s)\,G_2(s)}{1 + G_1(s)\,G_2(s)}$

08. The figure below gives two equivalent block diagrams



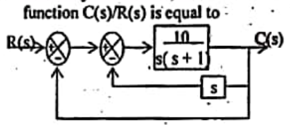
The value of transfer function of block marked 'X' is given by
(a) G (s) (b) 1 / G (s)
(c) 1 (d) 1 + G(s)

09. The figure shows two equivalent block diagrams



The transfer function of the block marked 'X' is given by
(a) G(s) (b) 1 / G(s)
(c) 1 (d) 1 + G(s)

10. For the system shown, the transfer function C(s)/R(s) is equal to



(a) $\frac{10}{s^2+s+10}$ (b) $\frac{10}{s^2+11s+10}$
(c) $\frac{10}{s^2+9s+10}$ (d) $\frac{10}{s^2+2s+10}$

Key for Objective Questions :

1. d 2. d 3. b 4. a 5. c
6. a 7. d 8. b 9. b 10. b

OBJECTIVE QUESTIONS

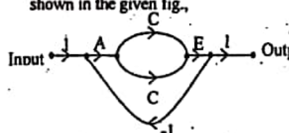
11. In a signal flow graph, the nodes represent
(a) the system variables
(b) the system gain
(c) the system parameters
(d) all the above

12. The branch of a signal flow graph represents
(a) the system variable
(b) the functional relations of the variables
(c) the system parameters
(d) none of the above

13. By applying Mason's gain formula, it is possible to get
(a) the ratio of the output variable to input variable only
(b) the system functional relations between any two variables
(c) the overall gain of the system
(d) the ratio of any variable to input variable only

14. Two or more loops in a signal flow graph are said to be non-touching
(a) if they do not have any common branch
(b) if they do not have any common loop
(c) if they have common node
(d) if they do not have any common node

15. The transfer function of the system shown in the given fig.,



(a) $\frac{ACE}{1+ACE}$ (b) $\frac{ACE}{1-ACE}$
(c) $\frac{2ACE}{1+ACE}$ (d) $\frac{2ACE}{1+2ACE}$

The time response has utmost importance for the design and analysis of control systems because these are inherently time domain systems where time is the independent variable. During the analysis of response, the variation of output with respect to time can be studied and it is known as time response. To obtain satisfactory performance of the system, the output behavior of the system with respect to time must be within the specified limits. From time response analysis and corresponding results, the stability of system, accuracy of system and complete evaluation can be studied very easily.

Due to the application of an excitation to a system, the response of the system is known as time response and it is a function of time. There are two parts of response of any system: (i) transient response and (ii) steady- state response.

Transient Response:

The part of the time response which goes to zero after large interval of time is known as transient response. In this case $\lim_{t=\infty} C(t)_t = 0$. From transient response, we get the following information:

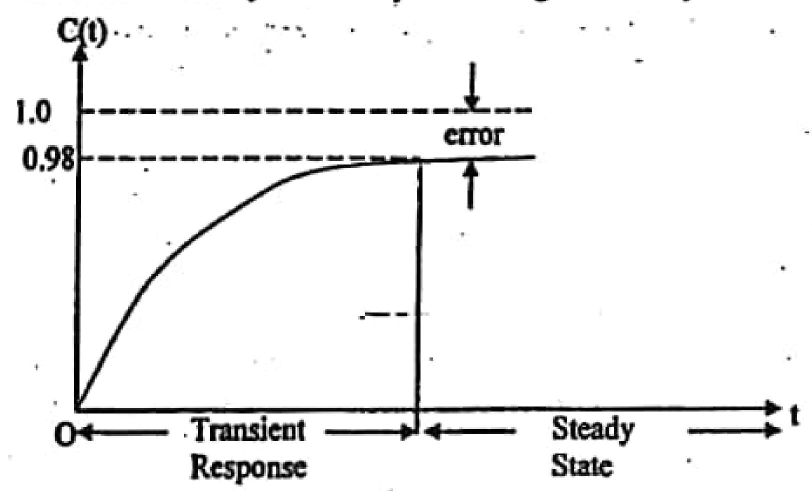
(a) The time interval after which the system responds taking the instant of application of excitation as reference.
(b) The total time that it takes to achieve the output for the first time.
(c) whether or not the output shoots beyond the desired value and how much.
(d) whether or not the output oscillates about its final value.
(e) The time that it takes to settle to the final value.

Steady State Response

The part of response that remains even after the transients have died out is said to be steady-state response. From steady – state response, we get the following information:
(a) The time that output takes to reach the steady – state
(b) Whether or not any error exists between the desired and the actual value.
(c) Whether this error is constant, zero, or infinite.
The total response of a system is the sum of transient response and steady – state response:

$$C(t) = C_{tr}(t) + C_{ss}$$

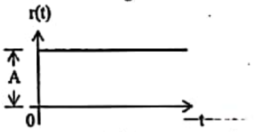
Figure shows the transient and steady – state responses along with steady – state error.



3.1Transiant analysis

Standard test signals:

(1) Step function : Step function is described as sudden application of input signal as illustrated in figure.

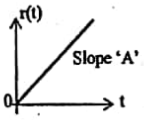


$r(t) = A\,u(t)$
$u(t) = 1 \; ; t \geq 0$
$= 0 \; ; t < 0$
In the Laplace transform form, $R(s) = A/s$

Step function /displacement function.

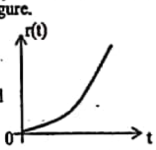
(2) Ramp function : The Ramp is a signal which starts at a value of zero and increases linearly with time. Mathematically,

$r(t) = At \; ; \text{for } t \geq 0$
$= 0 \; ; \text{for } t < 0$
In the Laplace transform from, $R(s) = A/s^2$
Ramp function is also called velocity function.



(3) Parabolic function : Parabolic function is described as more gradual application of input in comparison with ramp function as illustrated in figure.

$r(t) = At^2/2 \; ; \text{for } t \geq 0$
$r(t) = 0 \quad ; \text{for } t < 0$
If A =1, then $r(t) = t^2/2$ and the parabolic function is called unit parabolic function and the corresponding Laplace transform is



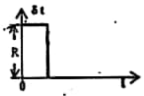
$R(s) = A/s^3$; Parabolic function is also called acceleration function

(4) Impulse function : A unit-impulse is defined as a signal which has zero value everywhere except at $t = 0$, where its magnitude is infinite. It is generally called the δ - function and has the following property : $\delta(t) = 0 \; ; t \neq 0$

$$\text{Unit impulse function} = \frac{d}{dt}(\text{unit step function})$$

Hence the Laplace transform of unit impulse function is derived from the Laplace transform of unit step function as follows

$\therefore £(\text{unit impulse function}) = s \cdot 1/s = 1$

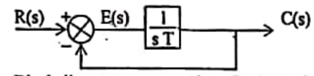


Time response of a First order Control System :

A first order control system is one wherein highest power of 's' in the denominator of its transfer function equals 1. Thus a first order control system is expressed by a transfer function given below :

$$\frac{C(s)}{R(s)} = \frac{1}{sT+1}$$

The block diagram representation of the above expression is shown in the below figure.



Block diagram representation of a first order control system.

Time response of a first order control system subjected to unit step input function :

The output for the system is expressed as

$$C(s) = R(s)\frac{1}{sT+1} \longrightarrow (1)$$

As the input is a unit step function $r(t) = 1$ and $R(s) = 1/s$

Therefore, substituting in Eq. (1) $C(s) = \frac{1}{s} \cdot \frac{1}{sT+1}$

Breaking R.H.S into partial fractions $C(s) = \frac{1}{s} - \frac{T}{sT+1}$

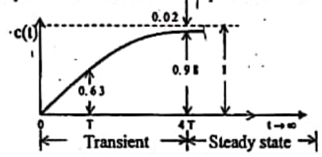
or $C(s) = \frac{1}{s} - \frac{1}{s+1/T}$ Taking Laplace transform on both sides

$$£^{-1}C(s) = £^{-1}\left[\frac{1}{s} - \frac{1}{s+1/T}\right] \quad \therefore \quad c(t) = 1 - e^{-t/T}$$

The error is given by $e(t) = r(t) - c(t) = 1 - (1 - e^{-t/T}) = e^{-t/T}$

The steady state error $= \lim_{t \to \infty} e^{-t/T} = 0$

The time response in relation to above equation is shown in the figure.



Time response of a first order control system subjected to step input

The graphical representation of the time response shown in figure indicates that the response is exponential type and the steady state value is 1 unit. As the time increases the disparity between the output and input approaches to nil, hence, the steady state error is zero.

Time response of a first order control system subjected to unit ramp input function :

The output for the system is expressed as $C(s) = R(s)\dfrac{1}{sT+1}$

As the input is a unit ramp function is $r(t) = t$ and $R(s) = 1/s^2$

Therefore $C(s) = \dfrac{1}{s^2}\dfrac{1}{sT+1}$ Breaking R.H.S. into partial fractions

$$C(s) = \frac{1-sT}{s^2} + \frac{T^2}{sT+1} \quad ; \quad C(s) = \frac{1}{s^2} - T\frac{1}{s} + T\frac{1}{s+1/T}$$

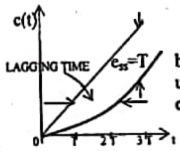
Taking inverse Laplace transform on both sides,

$$\pounds^{-1}C(s) = \pounds^{-1}\left(\frac{1}{s^2} - T\frac{1}{s} + T\frac{1}{s+1/T}\right); \quad c(t) = (t - T + Te^{-t/T})$$

The error is given by $e(t) = r(t) - c(t) = t - (t - T + Te^{-t/T}) = (T - Te^{-t/T})$

The steady state error is $e_{ss} = \lim_{t\to\infty}(T - Te^{-t/T}) = T$

The time response in relation to the above equation is shown in the figure.



The time response shown in figure indicates that during steady state, the output velocity matches with the input velocity but lags behind the input by time T and a positional error of T units exists in the system. It is also observed that lower the time constant lesser is the positional error and also lesser time lag.

Time response of a first order control system subjected to unit Impulse input function

The output for the system is expressed as $C(s) = R(s)\dfrac{1}{sT+1}$

As the input to the system is a unit impulse function, its Laplace transform is 1, i.e. $R(s) = 1$, therefore,

$$C(s) = 1.\frac{1}{sT+1}$$

Taking inverse Laplace transform on both sides of Eq.(2)

$$\pounds^{-1}C(s) = \pounds^{-1}\frac{1}{sT+1} \quad \text{or} \quad \pounds^{-1}C(s) = \pounds^{-1}(1/T).\frac{1}{s+1/T}$$

$$\therefore \quad c(t) = (1/T)e^{-t/T}$$

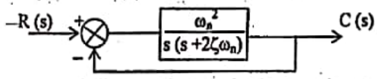
Time Response of Second Order control System :

A second order control system is one wherein the highest power of 's' in the denominator of its transfer function equals 2.

A general expression for the T.F of a second order control system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The block diagram representation of the transfer function given above is shown in the figure.



Block diagram of a second order control system

Characteristic Equation :

The general expression for the T.F. of a second order control system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The characteristic equation of a second order control system is given by

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The location of roots of the chara. equation for various values of ζ (keeping ω_n fixed) and the corresponding time response for a second order control system is shown in below figure.

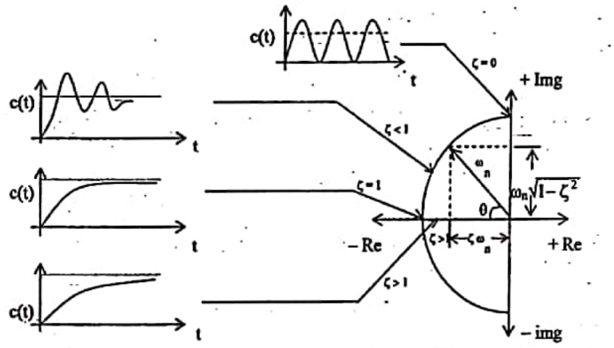


Fig : Location of roots of the characteristic equation and corresponding time response.

From above figure, it is inferred that the change over from underdamped to overdamped Response takes place at $\zeta = 1$. The value of ζ from the location of roots is calculated as

$$\boxed{\zeta = \cos\theta}$$

Time response of a second order control system subjected to unit step input function:

The output for the system is given by

$$C(s)=R(s).\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

As the input is a unit step function $r(t)=1$ and $R(s)=1/s$

Therefore, substituting in above Equation

$$C(s)=\frac{1}{s}.\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

The solution for the above equation

$$c(t)=1-\frac{\exp(-\zeta\omega_n t)}{\sqrt{1-\zeta^2}}\sin\left[(\omega_n\sqrt{1-\zeta^2})t+\left(\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right]$$

The time response expression is given by the above equation for values of $\zeta<1$ is, exponentially decaying oscillations having a frequency $\omega_n\sqrt{(1-\zeta^2)}$ and the time constant of exponential decay is $(1/\zeta\omega_n)$.

Where
- 'ω_n' is called natural frequency of oscillations.
- '$\omega_d=\omega_n\sqrt{(1-\zeta^2)}$' is called damped frequency of oscillations.
- 'ζ' affects the damping and called damping ratio.
- '$\zeta\omega_n$' is called damping factor or actual damping or damping coefficient.

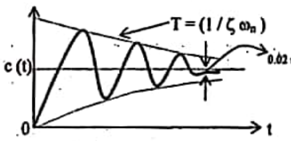


fig. (a)

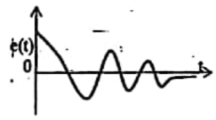


fig. (b)

Fig. Time response and error of a second order control system ($\zeta<1$, under damped case) subjected to unit step input function.

The time response of a second order control system is influenced by its damping ratio (ζ).The cases for the values of damping ratio as (a) $\zeta<1$ (b) $\zeta=0$ (c) $0<\zeta<1$ (d) $\zeta=1$ (d) $\zeta>1$ are considered below:

UNIT STEP RESPONSE ((0 <ζ <1), UNDERDAMPED)

As stated above, if $\zeta<1$ the time response presents damped oscillation and such a response is called underdamped response.

The response settles within 2% of the desired value (1 unit) after damping out the oscillations in a time $4T$, where $T=(1/\zeta\omega_n)$.

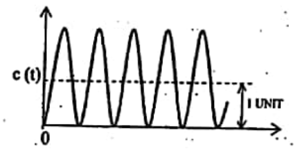
Unit step response when ($\zeta=0$, undamped system):

$$c(t)=1-\frac{\exp(-0\omega_n t)}{\sqrt{1-0^2}}\sin\left[\omega_n\sqrt{1-0^2}\,t+\tan^{-1}\left(\frac{\sqrt{1-0^2}}{0}\right)\right]$$

or $c(t)=1-\sin(\omega_n t+\tan^{-1}\infty)$ or $c(t)=1-\sin[\omega_n t+\pi/2]$

or $c(t)=(1-\cos\omega_n t)$

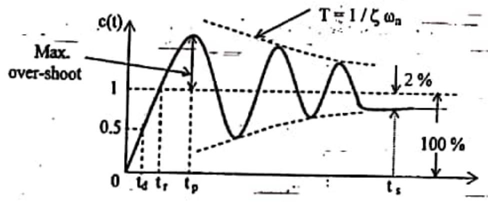
The time response to above equation is plotted in the below figure which indicates sustained (undamped) oscillations.



Unit step response when $0<\zeta<1$: (Under Damped systems only)

Transient response specification of second order control system:

The time response of an underdamped control system exhibits damped oscillations prior to reaching steady state. The specifications pertaining to time response during transient part are shown in the following figure.



(1) Delay Time: t_d

The time required for the response to rise from zero to 50% of the final value.

$$t_d=\frac{1+0.7\zeta}{\omega_d}$$

(2) The rise time : t_r

The rise time is the time needed for the response to reach from 10 to 90 % or 0 to 100 % of the desired value of the output at the very first instant. Usually 0 – 100 % basis is used for underdamped systems and 10 to 90 % for overdamped system.

$$t_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \quad ; \text{ where } \quad \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

(3) peak time: t_P

It is the time required for required for the response to rise zero to peaks of the time response $t_p = \pi / \omega_d$

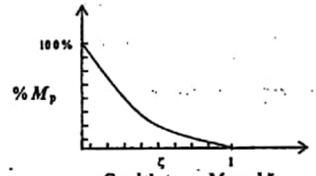
(4) Maximum overshoot : M_P

It gives the normalized difference between time response peak to steady state O/P

$$\text{Percentage } M_P = \frac{c(t)_{max} - C(\infty)}{C(\infty)} \times 100 = \frac{C(t)-1}{1} \times 100\%$$

$$\% M_p = \exp(-\zeta\pi/\sqrt{1-\zeta^2}) \times 100$$

A graph relating M_p and ζ is plotted in below figure.



Graph between M_p and ζ

(5) The Settling time : t_s

For 2 % tolerance band, the settling order time is given by

$$t_s = 4\,\frac{1}{\zeta\omega_n}$$

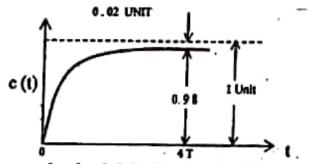
On 5 % basis the settling time for a second control system is given by

$$t_s = 3\,\frac{1}{\zeta\omega_n}$$

An expression for the time response of a second order control system having $\zeta = 1$ **(critically damped)** when subjected to a unit step input function is:

$$c(t) = [1 - \exp(-\zeta\omega_n t)(1 + \omega_n t)]$$

The time response in relation Eq. (11) is plotted in the below figure. The response is called critically damped response.



Time response of a second order C.S.(ζ =1 , critically damped) subjected to a unit step function.

An expression for the time response of a second order control system having $\zeta > 1$ **(Overdamped)** when subjected to unit step input function is derived hereunder :

- The output for the system is given by

$$C(s) = R(s)\,\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

As the input is a unit step function $r(t) = 1$ and $R(s) = 1/s$, therefore, substituting in the above equation $C(s) = (1/s).\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

It can also be written as $C(s) = (1/s).\dfrac{\omega_n^2}{(s+\zeta\omega_n)^2 - \omega_n^2(\zeta^2-1)}$

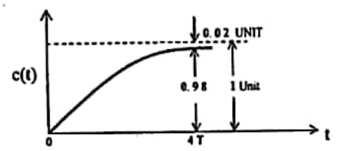
or $\quad C(s) = (1/s).\dfrac{\omega_n^2}{[s+(\zeta+\sqrt{\zeta^2-1})\,\omega_n]\,[s+(\zeta-\sqrt{\zeta^2-1})\,\omega_n]}$

Expanding R.H.S of above equation into partial fractions,

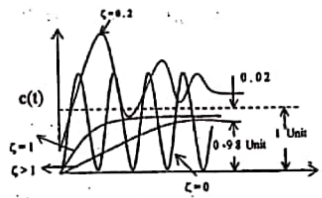
$$C(s) = \frac{1}{s} - \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta-\sqrt{\zeta^2-1})\,[s+(\zeta-\sqrt{\zeta^2-1})\,\omega_n]} + \frac{1}{2\sqrt{\zeta^2-1}\,(\zeta+\sqrt{\zeta^2-1})\,[s+(\zeta+\sqrt{\zeta^2-1})\,\omega_n]}$$

Taking inverse Laplace transform on both sides

$$c(t) = 1 - \frac{\exp[-(\zeta-\sqrt{\zeta^2-1})\,\omega_n t]}{2\sqrt{\zeta^2-1}\,(\zeta-\sqrt{\zeta^2-1})} + \frac{\exp[-(\zeta+\sqrt{\zeta^2-1})\,\omega_n t]}{2\sqrt{\zeta^2-1}\,(\zeta+\sqrt{\zeta^2-1})}$$



Time res. of a second order C.S.($\zeta > 1$, over-damped) subjected to a unit step input function.



Comparison of unit step input time response of a second order control system for different values of 'ζ'.

OBJECTIVE QUESTIONS

01. The radial distance between a pole and the origin gives
    a) damped frequency of oscillation.
    b) undamped frequency of oscillation.
    c) time constant
    d) natural frequency of oscillation.

02. For a type 1, second order control system, when there is an increase of 25 % in its natural frequency, the steady-state error to unit ramp input is
    a) increased by 20 % of its value.
    b) equal to $2\zeta/\omega_n$, where ζ = damping factor.
    c) decreased by 21 %
    d) decreased effectively by 20 %

03. In a type 1, second order system, first peak overshoot occurs at a time equal to

    (a) $\frac{\pi\omega_n}{\sqrt{1-\zeta^2}}$ (b) $\frac{\omega_n}{\sqrt{1-\zeta^2}}$

    (a) $\frac{\pi\omega_n}{\sqrt{1+\zeta^2}}$ (d) $\frac{\pi/\omega_n}{\sqrt{1-\zeta^2}}$

04. Type number of a system gets decreased if
    a) first an integrator and then a differentiator is included in the system.
    b) an integrator is included in the forward path.
    c) a differentiator is included in a parallel path.
    d) a differentiator is included in the forward path.

05. When the pole of a system is moved towards the imaginary axis, then
    a) settling time decreases.
    b) settling time increases by 20 % of initial value.
    c) steady-state error is reduced to zero.
    d) settling time of the system increases.

06. The damping factor of a second order system whose response to unit step input is having sustained oscillations is
    a) = 1 b) > 1
    c) < 1 d) = 0

07. The transient response of a system with feedback when compared to that without feedback
    a) decays slowly.
    b) rises slowly.
    c) rises more quickly.
    d) decays more quickly.

08. The settling time for the system $G(s) = \frac{(s+3)}{s^2+5s+25}$ is .... seconds when the output settles within ± 2 % for a unit step input.
    a) 0.8 b) 1.2
    c) 2.0 d) 1.6

09. The type of the system whose transfer function is given by

    $G(s) = \frac{(s+3)}{s^5+s^4+s^3+3s^2+2s}$ is

    (a) 3 (b) 2 (c) 5 (d) 1

10. Physically the damping ratio represents the
    a) energy available for transfer.
    b) energy available for exchange.
    c) ratio of energy available for exchange to that available for transfer.
    d) ratio of energy lost to the energy available for exchange.

11. The static acceleration constant of a type 2 system is
    a) infinite b) zero
    c) cannot be found out d) finite.

12. The time domain specification which is dependent only on, the damping factor is
a) rise time b) peak time
c) setting time d) peak overshoot.

Key for Objective Questions :

1 to 12 .... (d)

3.2 Steady state Analysis :

The steady state part of time response reveals the accuracy of a control system. Steady state error is observed if the actual output does not exactly match with the input.

$$e(t) = r(t) - c(t)$$

Steady state error, $e_{ss} = \underset{t \to \infty}{Lt}\ e(t)$

$$= \underset{t \to \infty}{Lt}\ [r(t) - c(t)]$$

Using final value theorem,

$$e_{ss} = \underset{s \to 0}{Lt}\ s\,E(s)$$

$$C(s) = E(s)\,G(s) \Rightarrow E(s) = \frac{C(s)}{G(s)}$$

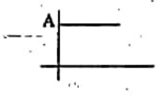
$$\frac{C(s)}{G(s)} = \frac{R(s)}{1+G(s)}$$

$$\therefore e_{ss} = \underset{s \to 0}{Lt}\ s . \frac{R(s)}{1+G(s)}$$

The open-loop transfer function, the type indicates the number of poles at the origin and the order indicates the total number of poles. The type of the system determines steady state response and the order of the system determines transient response.

Standard test signals used in Steady state response :

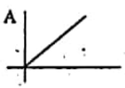
(1) Step input signal :



$r(t) = A\,u(t)$

$R(s) = A/s$

(2) Ramp input signal :



$r(t) = At$

$R(s) = A/s^2$

(3) Parabolic input signal :

$r(t) = At^2/2$

$R(s) = A/s^3$

For step input

$$e_{ss} = \underset{s \to 0}{Lt}\ s . \frac{R(s)}{1+G(s)}$$

$$= \underset{s \to 0}{Lt}\ s . \frac{A/s}{1+G(s)} = \underset{s \to 0}{Lt}\ \frac{A}{1+G(s)} = \frac{A}{1+K_p}$$

where $K_p = \underset{s \to 0}{Lt}\ G(s)$ = Position error constant

For ramp input

$$e_{ss} = \underset{s \to 0}{Lt}\ s . \frac{R(s)}{1+G(s)}$$

$$= \underset{s \to 0}{Lt}\ s . \frac{A/s^2}{1+G(s)} = \underset{s \to 0}{Lt}\ \frac{A}{s[1+G(s)]} = \frac{A}{K_v}$$

where $K_v = \underset{s \to 0}{Lt}\ s\,G(s)$ = Velocity error constant

For parabolic input

$$e_{ss} = \underset{s \to 0}{Lt}\ s . \frac{R(s)}{1+G(s)}$$

$$= \underset{s \to 0}{Lt}\ s . \frac{A/s^3}{1+G(s)} = \underset{s \to 0}{Lt}\ \frac{A}{s^2[1+G(s)]} = \frac{A}{K_a}$$

where $K_a = \underset{s \to 0}{Lt}\ s^2 G(s)$ = Acceleration error constant

	Type 0	1	2	3
Step	$\frac{A}{1+K_p}$	0	0	0
Ramp	∞	A/K_v	0	0
Parabolic	∞	∞	A/K_a	0

OBJECTIVE QUESTIONS

01. The presence of non-linearities in a control system tends to introduce

a) transient error b) instability
c) static error d) steady-state error

02. The static acceleration constant of a type 2 system is

a) infinite b) zero
c) cannot be found out d) finite

03. The transfer function of the system which will have more steady state error for step input is

(a) $\frac{80}{(s+1)(s+2)(s+3)}$

(b) $\frac{120}{s(s+1)(s+15)}$

(c) $\frac{60}{(s+0.5)(s+3)(s+5.5)}$

(d) $\frac{120}{(s+1)(s+4)(s+15)}$

04. The presence or absence of steady-state error for any given system depends upon

a) presence or absence of pole at the infinity.
b) presence or absence of poles and zeros at the origin.
c) absence or presence of zeros at the origin.
d) absence or presence of pole at the origin.

05. When the gain 'k' of a system is increased, the steady-state error of the system

a) increases.
b) remains unchanged
c) may increase or decrease.
d) decreases.

06. The plant is represented by the transfer function.The system is given a degenerative feedback. The effective of the feedback is to shift the pole

a) positively to $s=(a+k)$ and reduce the time constant to $\alpha+(1/k)$
b) negatively to $s=-(\alpha+k)$ and increase the time constant to $\alpha+k$.
(c) negatively to $s=-(\alpha+k)$ and reduce the time constant to $\alpha+(1/k)$.
(d) negatively to $s=-(\alpha+k)$ and decrease the time constant to $[1/(\alpha+k)]$.

07. In a system with input R (s) and output C (s), the transfer functions of the plant and the feedback system is given by G (s) and H (s) respectively. The system has got a negative feedback. Then the error signal is given by the expression :

(a) $E(s)=\frac{G(s)R(s)}{1+G(s)H(s)}$

(b) $E(s)=C(s)G(s)$

(c) $E(s)=\frac{1}{1+G(s)H(s)}$

(d) $E(s)=\frac{R(s)}{1+G(s)H(s)}$

08. The static error constants depends on

a) the order of the system
b) the type of the system
c) both type and order of the system
d) None of the above

Key :

1. d 2. d 3. d 4. d 5. d

6. d 7. d 8. b 9. d

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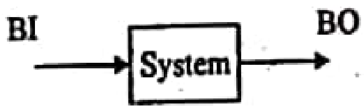
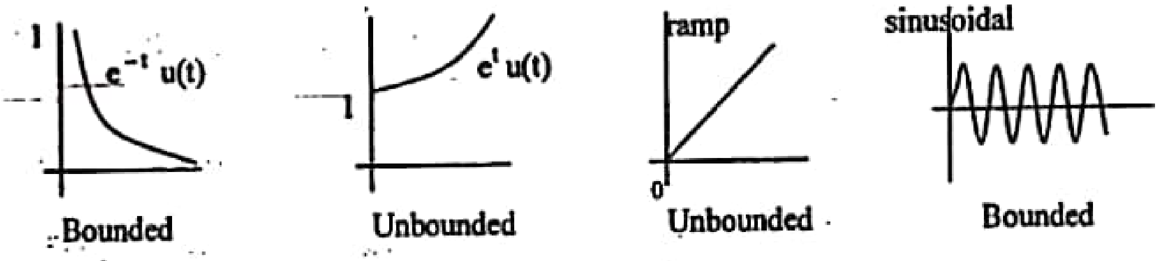
Concept of stability:

Any system is called as a *stable system* if the output of the system is bounded for a bounded input. Any signal is called bounded if the max. and min. value are finite.



Eg :



- Stability of any system depends only on the location of poles but not on the location of zeros.
- If the poles are located in left side of s – plane, then the system is stable.
- If the roots are located on imaginary axis including the origin (except repeated roots), the system is stable.
- If the poles are located in right half of s – plane, then the system is unstable.
- As pole is approaches origin, stability decreases.
- When roots are located on imaginary axis, then the system is marginally stable.
- The poles which are close to the origin are called dominant poles.
- The systems are classified as
  1) Absolutely stable systems
  2) Unstable systems
  3) Conditionally stable systems
- When variable parameter is varied from 0 to ∞, if the poles are located on left side and it is always stable, then it is absolutely stable.
- When variable parameter is varied and a system is stable for values 0 to ∞, at some point onwards there is (are) pole(s) in right side then it is called conditionally stable.
- Techniques used to calculate stability are
  1) Routh-Hurwitz criterion
  2) Root locus
  3) Bode plot
  4) Nyquist plot
  5) Nicholas chart

4.2 ROOT LOCUS TECHNIQUE

It is the graphical representation of the roots of the characteristic equation, then the variable parameter is varied from 0 to ∞.

1) Root Locus (RL)	($K \rightarrow 0$ to ∞)
2) Complementary RL	($K \rightarrow 0$ to $-\infty$)
3) Complete RL	($K \rightarrow -\infty$ to ∞)
4) Root contour	(Multiple parameter variation)

Concept of Root locus :

It is not possible to plot the root locus if there is no variable parameter in characteristic equation.

Classification of stable systems :

1) Undamped system	(roots on imaginary axis i.e., real part = 0)
2) Under damped system	(imaginary but real part is negative)
3) Critically damped	(roots are real and same)
4) Over damped system	(roots are real and different)

Rules for the construction of Root locus :

- The root locus is always symmetrical with respect to the real axis.
- The root locus always starts (K= 0) from the open - loop poles and terminates (K= ∞) on either finite open – loop zeros or infinity. This statement is valid only if P = Z.
- The number of separate branches of the root locus equals either the number of open – loop poles or number of open – loop zeros whichever is greater.

$$N = P, \text{ if } P > Z$$
$$N = Z, \text{ if } Z > P$$

- A section of root locus lies on the real axis if the total number of open – loop poles and zeros to the right of the section is odd.
- The value of 'K' at any point on the root locus can be calculated by using the magnitude criteria.

$$K = \frac{\text{Product of poles magnitude (or length)}}{\text{Product of zeros magnitude (or length)}}$$

- If $P \neq Z$, some of the branches terminate at '∞' or some of the branches will start from '∞'.

  If $P > Z$, $(P - Z)$ branches will terminate at '∞'.
  If $Z > P$, $(Z - P)$ branches will start from '∞'.

→ Whenever any branch will terminate at '∞' means that a zero is located at '∞'.
→ Whenever any branch is start from '∞' means that a pole is located at '∞'.

- **The angle of asymptotes :**

  If $P > Z$, $(P - Z)$ branches will terminate at '∞' along straight line asymptotes whose angles are :

$$\frac{(2q+1)\,180°}{P-Z}$$

  If $Z > P$, $(Z - P)$ branches will start from '∞' along straight line asymptotes whose angles are

$$\frac{(2q+1)\,180°}{Z-P}$$

Centroid : The asymptotes meet the real axis at centroid.

$$\sigma = \frac{\text{Sum of real parts of poles} - \text{Sum of real parts of zeros}}{P - Z}$$

- **Intersection points with imaginary axis :** The value of 'K' and the point at which the Root locus branch crosses the imaginary axis is determined by applying Routh criterion to the characteristic equation. The roots at the intersection point are imaginary.
- **Break – away point and break – in point :**

  Break – away point is calculated when root locus lies between two poles.
  Break – in point is calculated when root locus lies between two zeros.

  Break – away or break – in point is calculated by solving $\boxed{\frac{dK}{ds} = 0}$

Procedure :

a) From the characteristic equation (C.E.)
b) Rewrite the characteristic equation in the form of $K = f(s)$
c) $dK / ds = 0$
d) The root of $dk / ds = 0$ gives the valid and invalid break point
c) The valid break point which must be on root locus branch

- **Angle of arrival :**

  It is applied when there are complex zeros.

  $\boxed{\phi_A = 180° + \phi}$

  where $\phi = \angle\text{poles} - \angle\text{zeros}$

- **Angle of departure :**

  It is applied when there are complex poles.

  $\boxed{\phi_D = 180° - \phi}$

Complementary Root locus :

In this the magnitude criteria remains same but angle criteria changes.

i.e., $\angle$Zeros $-$ $\angle$ Poles = even multiples of π

1) Asymptotic angles $= \dfrac{(2q)\ 180^\circ}{P-Z}$

2) Angle of departure $= 180^\circ - \phi$

where $\phi = \angle$poles $-$ $\angle$ zeros

3) Angle of arrival $= 180^\circ + \phi$

4) A point on the real axis lies in the complementary RL, if the number of poles and zeros to the right side of any point is a even number.

Example : Sketch the complete root locus for the system having

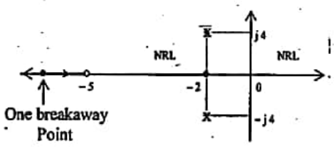
$$G(s)\,H(s) = \frac{K\,(s+5)}{(s^2+4s+20)}$$

Sol : ***Step 1 :*** Number of poles $P = 2$, $Z = 1$, $N = P - Z$

One branch has to terminate at finite zero $s = -5$ while $P - Z = 1$ branch has to terminate at ∞.

Starting points of branches are, $-2 \pm j\,4$.

Step 2 : Pole-zero plot of the system is shown below.



Step 3 : Angle of asymptotes

$$\theta = \frac{(2q+1)\ 180^0}{P-Z}, \qquad q = 0$$

Step 4 : Centroid.

As there is one branch approaching to ∞ and one asymptote exists, centroid is not required.

Step 5 : Breakaway point.

$$1 + G(s)\,H(s) = 0$$

$$\therefore \quad s^2 + 4s + 20 + K\,(s+5) = 0$$

$$\frac{dK}{ds} = 0 \quad \Rightarrow -s\,(s+10) = 0$$

$s = 0$ and $s = -10$ are breakaway points. But $s = 0$ cannot be breakaway point. Hence $s = -10$ is valid breakaway point.

Step 6 : Intersection with imaginary axis.

Characteristic equation,

$$s^2 + 4s + 20 + K\,(s+5) = 0$$

$$s^2 + s\,(K+4) + (20+5K) = 0$$

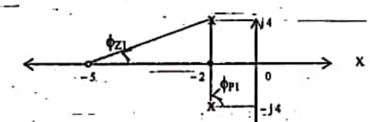
Routh's Array can be formed as below :

s^2	1	$20 + 5K$
s^1	$K + 4$	0
s^0	$20 + 5K$	

$K_{mar} = -4$ makes s^1 row as row of zeros.

But as it is negative, there is no intersection of root locus with imaginary axis.

Step 7 : Angle of departure.

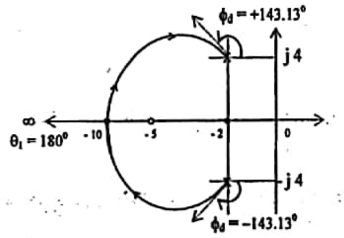


$\phi_{P1} = 90^0$, $\quad \phi_{Z1} = \tan^{-1}(4/3) = 53.13^0$

$\therefore \quad \phi = \Sigma\,\phi_P - \Sigma\,\phi_Z = 36.86^0$

$\therefore \quad \phi_d = 180^0 - \phi = +143.13^0 \qquad$ at $-2 + j\,4$ pole.

$\phi_d = -143.13^0 \qquad$ at $-2 - j\,4$ pole.



Step 9 **: Prediction of stability**

For all ranges of K i.e., $0 < K < \infty$, both the roots are always in left half of s-plane. So system is inherently stable.

Example : Sketch the complete root locus of system having

$$G(s)\,H(s) = \frac{K}{s\,(s+1)\,(s+2)\,(s+3)}$$

Sol. : ***Step 1*** **:** P = 4, Z = 0 & N = 4 i.e., four branches in the root locus.

Step 2 **:** All four branches starts from open-loop poles and terminates at ∞.

Step 3 **:** Angle of asymptotes $= \frac{(2q+1)\,180°}{4} = 45°, 135°, 225°, 315°$

Step 4 **:** Centroid $= \frac{0-1-2-3}{4} = -1.5$

Step 5 **:** Breakaway point

$$K = -s^4 - 6s^3 - 11s^2 - 6s$$

$$\frac{dK}{ds} = 0 \Rightarrow s = -1.5, -0.381, -2.619$$

Here, – 1.5 lies in the root locus and – 0.381, & 2.619 lies in the complementary root locus.

Step 6 **:** Intersection of root locus imaginary axis.

Characteristic Equation $s^4 + 6s^3 + 11s^2 + 6s + K = 0$

s^4	1	11	K
s^3	6	6	0
s^2	10	K	0
s^1	(60 – 6K) / 10	0	
s^0	K		

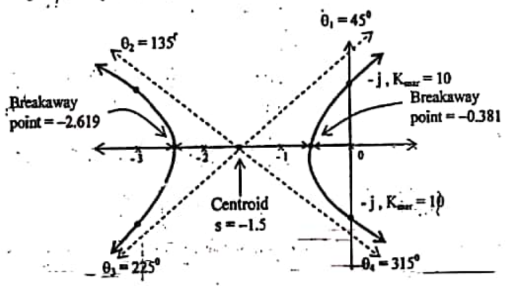
$\therefore \quad 60 - 6K = 0 \qquad \therefore K_{mar} = +10$

Auxiliary equation :

$$10s^2 + K = 0$$

At K = 10, $s^2 = -1$, $s = \pm j$

Step 7 **:** Complete root locus.



Step 8 **:** For 0 < K < 10, system is absolutely stable. At K = 10, system is marginally stable oscillating with 1 rad/sec. For K > 10, system is unstable.

Complementary Root Locus :

Step 1 : $P = 4, Z = 0$ & $N = 4$ i.e., four branches in the root locus.

Step 2 : All four branches starts from open-loop poles and terminates at ∞.

Step 3 : Angle of asymptotes $= \dfrac{(2q)\,180^0}{4} = 0^0, 90^0, 180^0, 270^0$

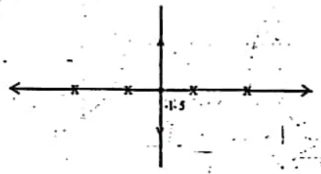
Step 4 : Centroid $= \dfrac{0-1-2-3}{4} = -1.5$

Step 5 : Breakaway point

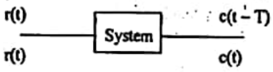
$$K = -s^4 - 6s^3 - 11s^2 - 6s$$

$$\frac{dK}{ds} = 0 \quad \Rightarrow s = -1.5, -0.381, -2.619$$

Here, -1.5 lies in the root locus and -0.381, & 2.619 lies in the complementary root locus.

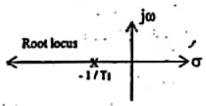
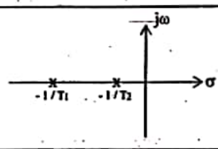
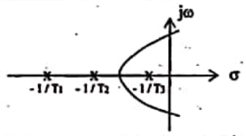
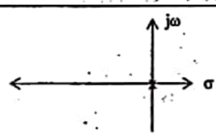
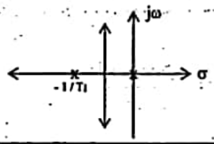
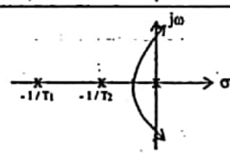


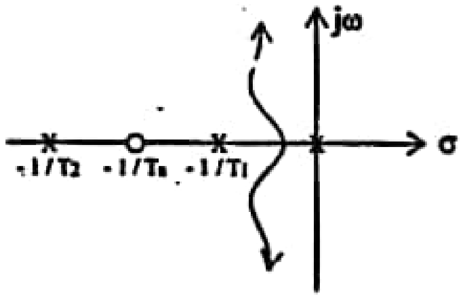
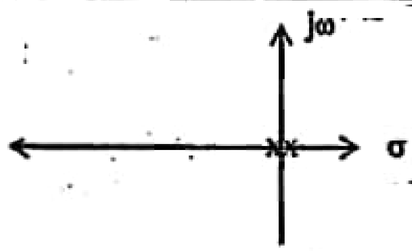
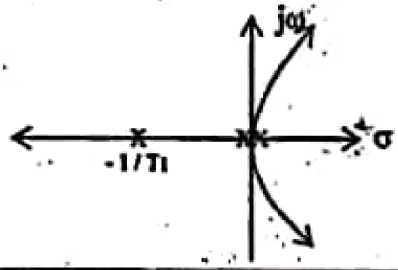
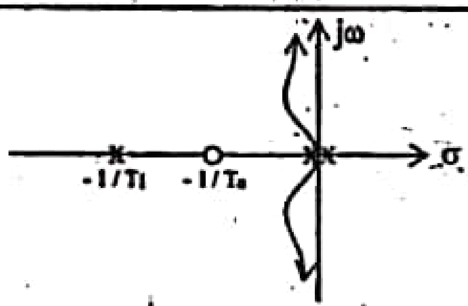
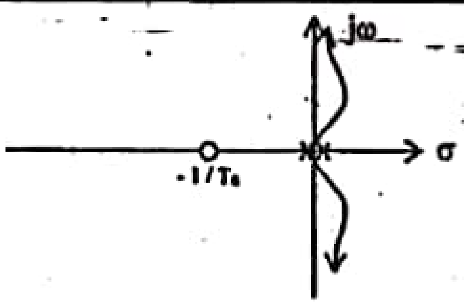
RL of system with transportation lag :



$$\text{Transfer function} = \frac{L[\text{Output}]}{L[\text{Input}]} = \frac{C(s)\,e^{-sT}}{R(s)}$$

Root Locus Plots for Typical Transfer Functions :

G(s)	Root Locus
1. $\dfrac{K}{sT_1 + 1}$	
2. $\dfrac{K}{(sT_1+1)(sT_2+1)}$	
3. $\dfrac{K}{(sT_1+1)(sT_2+1)(sT_3+1)}$	
4. $\dfrac{K}{s}$	
5. $\dfrac{K}{s(sT_1+1)}$	
6. $\dfrac{K}{s(sT_1+1)(sT_2+1)}$	

7. $\dfrac{K(sT_a+1)}{s(sT_1+1)(sT_2+1)}$	
8. $\dfrac{K}{s^2}$	
9. $\dfrac{K}{s^2(sT_1+1)}$	
10. $\dfrac{K(sT_a+1)}{s^2(sT_1+1)}$	
11. $\dfrac{K(sT_a+1)}{s^3}$	

The various frequency response analysis techniques are

1) Bode plot
2) Polar plot
3) Nyquist plot
4) M & N circles
5) Nicholas chart

1) Bode plots :

It is used to draw the frequency response of a open loop and closed-loop system. The representation of the logarithm of $|G(j\omega)|$ and phase angle of $G(j\omega)$, both plotted against frequency in logarithmic scale. These plots are called Bode plots.

Bode Plot of first order system :

Let the Transfer Function $= \dfrac{1}{1+Ts}$

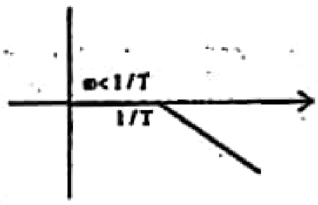
subs. $s = j\omega$

$$\text{T.F.} = \frac{1}{1+j\omega T}$$

$$M = \frac{1}{\sqrt{1+(\omega T)^2}} \quad ; \qquad \phi = -\tan^{-1}(\omega T)$$

$$M = 20\log\frac{1}{\sqrt{1+(\omega T)^2}} = -10\log[1+(\omega T)^2]$$

$\omega << 1/T$ $\omega << 1/T$



ω

$M_{dB} \cong 10\log 1$ $M_{dB} = -10\log(\omega T)^2$

$\cong 0$ $= -20\log\omega T$

Therefore, the error at the corner frequency $\omega = 1/T$ is

$$-10\log 2 + 10\log 1 = -3\text{ dB}$$

The error at frequency ($\omega = 1/2T$) one octave below the corner frequency is

$$-10\log(1+\tfrac{1}{4}) + 10\log 1 = -1\text{ Db}$$

Bode Plot of second order system :

$$T.F. = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

subst. $s = j\omega$

$$T.F. = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n \omega + \omega_n^2}$$

Divide with 'ω_n^2'

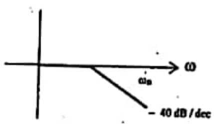
$$= \frac{1}{(1-\mu^2)^2 + j2\zeta\mu} \qquad [\mu = \omega/\omega_n]$$

$$\therefore \quad M = \frac{1}{\sqrt{(1-\mu^2)^2 + (2\zeta\mu)^2}} \quad ; \quad \phi = -\tan^{-1}\left(\frac{2\zeta\mu}{1-\mu^2}\right)$$

$$M_{dB} = -10\log\left[(1-\mu^2)^2 + 4\zeta^2\mu^2\right]$$

Case 1) When $\mu < 1 \Rightarrow (\omega/\omega_n) < 1 \Rightarrow \omega < \omega_n$

$$M_{dB} \cong -10 \log 1 \cong 0 \text{ dB}$$



Case 2) When $\mu > 1 \Rightarrow (\omega/\omega_n) > 1 \Rightarrow \omega > \omega_n$

$$M_{dB} \cong -10 \log \mu^4 \cong -40 \log \mu$$

The error between the actual magnitude and the asymptotic approximation is as given below.

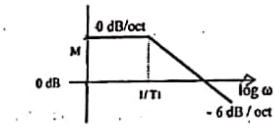
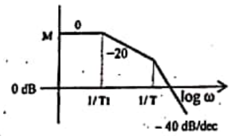
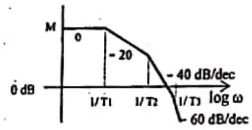
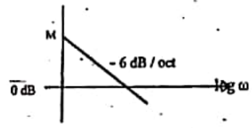
For ~~$0 < \mu << 1$, the error~~ is

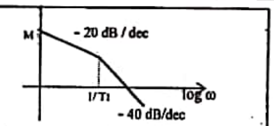
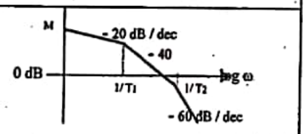
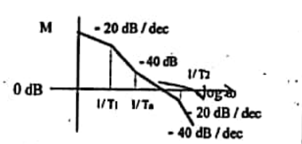
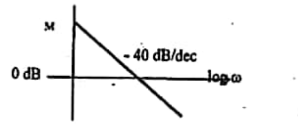
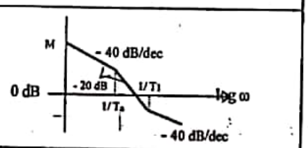
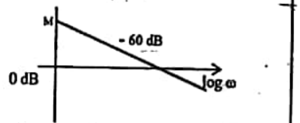
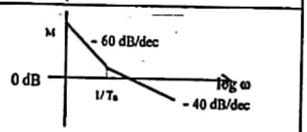
$$-10\log[(1-\mu^2)^2 + 4\zeta^2\mu^2] + 10\log 1$$

and for $1 < \mu << \infty$, the error is

$$-10\log[(1-\mu^2)^2 + 4\zeta^2\mu^2] + 40\log\mu$$

Bode Plots for Typical Transfer Functions :

G(s)	Bode Plot
1. $\frac{K}{sT_1 + 1}$	
2. $\frac{K}{(sT_1+1)(sT_2+1)}$	
3. $\frac{K}{(sT_1+1)(sT_2+1)(sT_3+1)}$	
4. $\frac{K}{s}$	

5. $\dfrac{K}{s(sT_1+1)}$	
6. $\dfrac{K}{s(sT_1+1)(sT_2+1)}$	
7. $\dfrac{K(sT_a+1)}{s(sT_1+1)(sT_2+1)}$	
8. $\dfrac{K}{s^2}$	
9. $\dfrac{K(sT_a+1)}{s^2(sT_1+1)}$	
10. $\dfrac{K}{s^3}$	
11. $\dfrac{K(sT_a+1)}{s^3}$	

2) Polar plot :

The sinusoidal transfer function $G(j\omega)$ is a complex function and is given by

$$G(j\omega) = Re[G(j\omega)] + j\, I_m[G(j\omega)]$$

or

$$G(j\omega) = |G(j\omega)| \angle G(j\omega) = M \angle \phi$$

from above equation, it is seen that $G(j\omega)$ may be represented as a phasor of magnitude M and phase angle ϕ. As the input frequency ω is varied from 0 to ∞, the magnitude M and phase angle ϕ change and hence the tip of the phasor $G(j\omega)$ traces a locus in the complex plane. The locus thus obtained is known as *polar plot*.

When a transfer function consists of 'P' poles and 'Z' zeros, and it doesn't consist poles at origin then the polar plot starts from 0^0 with some magnitude and terminates at $-90^0 \times (P-Z)$ with zero magnitude.

When a transfer function consists of poles at origin, then the polar plot starts from $-90^0 \times$ no. of poles at origin with '∞' magnitude and ends at $-90^0 \times (P-Z)$ with zero magnitude.

2) Nyquist Stability Criteria :

It is used to determine the stability of a closed-loop system using polar plots. This concept is derived from complex analysis using 'Principle of Argument'.

Let $$G(s) = \frac{(s+Z_1)(s+Z_2)}{(s+P_1)(s+P_2)} \longrightarrow (1)$$

Characteristic Equation, i.e, $1 + G(s)$

$$1 + G(s) = 1 + \frac{(s+Z_1)(s+Z_2)}{(s+P_1)(s+P_2)}$$

$$= \frac{(s+P_1)(s+P_2) + (s+Z_1)(s+Z_2)}{(s+P_1)(s+P_2)} \longrightarrow (2)$$

From (1) and (2), the open-loop poles and CE poles are same.

$$C.E. = \frac{(s+Z_1^1)(s+Z_2^1)}{(s+P_1)(s+P_2)} \longrightarrow (3)$$

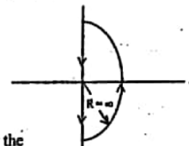
$$\text{Overall transfer function} = \frac{G(s)}{1+G(s)} = \frac{(s+Z_1)(s+Z_2)}{(s+Z_1^1)(s+Z_2^1)} \longrightarrow (4)$$

From (3) and (4), the C.E zeros and closed-loop poles are same.

→For the closed-loop system to be stable, the zeros of the C.E should not be located on the right half of the s-plane.

Using Principle of Argument

$$Q(s) = 1 + G(s)$$



Consider a contour as shown which covers the entire right half of the s-plane. If each and every point is along the boundary of the contour is substituted in C.E according to the principle of argument.

The no. of encircles with respect to origin, $N = Z - P$

where Z and P are the zeros and poles of the C.E located inside the contour or located in right half of the s-plane.

For the closed-loop system to be stable, $Z = 0$.

→For the open-loop system to be stable, $P = 0$, then $N = Z$.

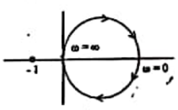
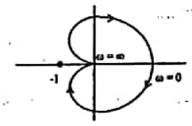
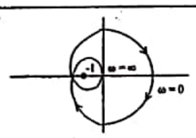
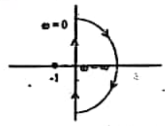
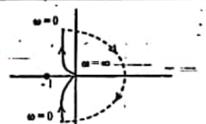
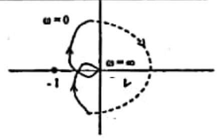
In $N = Z - P$, Z becomes '0' only if $N = 0$ [q(s) contour shouldn't encircle the origin]

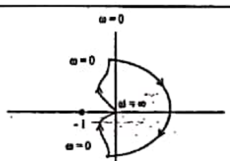
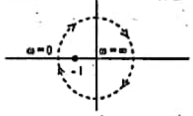
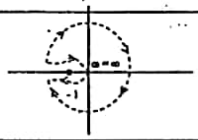
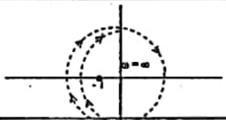
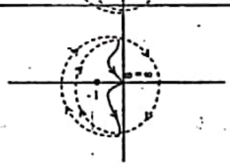
If the open-loop system is stable, the closed-loop system will be stable only if the Nyquist contour doesn't encircles origin.

→For the open-loop system to be unstable, $P \neq 0$.

If the open-loop system is unstable, the closed-loop system will be stable only if the Nyquist contour encircles origin in clockwise direction. The number of encirclements should be equal to the number of open-loop poles located inside the contour.

Nyquist Plots for Typical Transfer Functions :

G(s)	Nyquist Plot
1. $\frac{K}{sT_1+1}$	
2. $\frac{K}{(sT_1+1)(sT_2+1)}$	
3. $\frac{K}{(sT_1+1)(sT_2+1)(sT_3+1)}$	
4. $\frac{K}{s}$	
5. $\frac{K}{s(sT_1+1)}$	
6. $\frac{K}{s(sT_1+1)(sT_2+1)}$	

7. $\dfrac{K(sT_a+1)}{s(sT_1+1)(sT_2+1)}$	
8. $\dfrac{K}{s^2}$	
9. $\dfrac{K(sT_a+1)}{s^2(sT_1+1)}$	
10. $\dfrac{K}{s^3}$	
11. $\dfrac{K(sT_a+1)}{s^3}$	

4) M & N Circles

Constant Magnitude Loci : M-Circles

M – circles are used to determine the magnitude response of a closed-loop system using open-loop transfer function.

It is applicable only for unity feedback system. The open-loop transfer function $G(j\omega)$ of a unity feedback control system is a complex quantity and can be expressed as

$$G(j\omega).1 = x + jy$$

Since $$M = \frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)}$$

$$\therefore \quad M = \frac{x+jy}{1+x+jy}$$

$$\therefore \quad M = \frac{\sqrt{x^2+y^2}}{\sqrt{(1+x)^2+y^2}}$$

On squaring on both sides and simplifying following equation is obtained :

$$(1-M^2)x^2 - 2M^2x + (1-M^2)y^2 = M^2$$

or $$x^2 - \frac{2M^2}{(1-M^2)}x + y^2 = \frac{M^2}{(1-M^2)}$$

Add $\left(\dfrac{M^2}{1-M^2}\right)^2$ to both sides ,

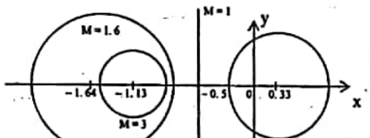
$$\therefore \quad x^2 - \frac{2M^2}{(1-M^2)}x + \left(\frac{M^2}{1-M^2}\right)^2 + y^2 = \frac{M^2}{(1-M^2)} + \left(\frac{M^2}{1-M^2}\right)^2$$

or $$\left(x - \frac{M^2}{1-M^2}\right)^2 + y^2 = \left(\frac{M}{1-M^2}\right)^2$$

For different values of M, above Eq. represents a family of circles with centre at $x = (M^2/1-M^2)$, $y = 0$ and radius as $(M^2/1-M^2)$. On a particular circle the value of M (magnitude of closed-loop transfer function) is constant, therefore, these circles are called M-circles.

The centres and radii of M-circles for different values of M are given in the following table and M-circles are drawn in the following figure.

M	centre $x = M^2/1-M^2, y=0$	Radius $r = M/1-M^2$
0.5	0.33	0.67
1.0	∞	∞
1.2	-3.27	2.73
1.6	-1.64	1.03
2.0	-1.33	0.67



In G (jω) plane the Nyquist plot is superimposed on M-circle and the points of intersection that gives the magnitude of C (jω) / R (jω) at different values of 'ω'.

Constant Phase Angles Loci : N-circles:

N – circles are used to determine the phase response of a closed-loop system using open-loop transfer function.

The phase angle of the closed-loop transfer function of a unity feedback system is given by

$$\angle\frac{C(j\omega)}{R(j\omega)} = \angle\frac{x+jy}{1+x+xy}$$

The phase angle is denoted by ϕ, therefore,

$$\phi = \tan^{-1}(y/x) - \tan^{-1}[y/(1+x)]$$

or $$\tan\phi = \tan\left\{\tan^{-1}(y/x) - \tan^{-1}[y/(1+x)]\right\}$$

$$= \frac{\tan\left[\tan^{-1}(y/x)\right] - \tan\left[\tan^{-1}[y/(1+x)]\right]}{1+\tan\left[\tan^{-1}(y/x)\right].\tan\left[\tan^{-1}[y/(1+x)]\right]}$$

$$= \frac{(y/x)-[y/(1+x)]}{1+(y/x).[y/(1+x)]}$$

or $$\tan\phi = \frac{y}{x^2+x+y^2}$$

Substituting $\tan\phi = N$ in above equation

$$\therefore \quad N = \frac{y}{x^2+x+y^2}$$

or $$x^2+x+y^2-(y/N) = 0$$

Add $\left(\frac{1}{4}+\frac{1}{4N^2}\right)$ on both sides

$$\therefore \quad \left[x^2+x+\frac{1}{4}\right]+\left[y^2-\frac{y}{N}+\frac{1}{4N^2}\right] = \left(\frac{1}{4}+\frac{1}{4N^2}\right)$$

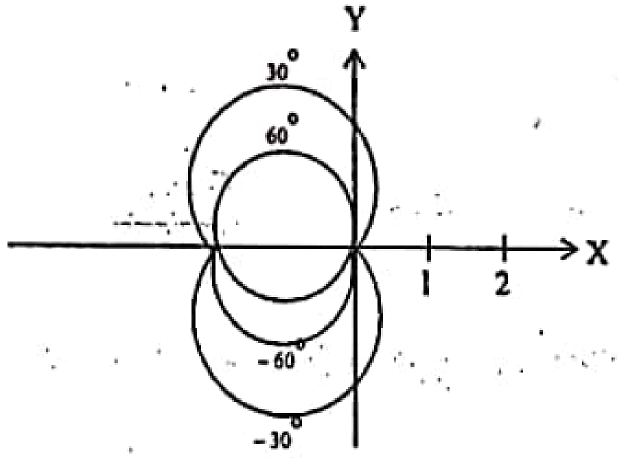
or $$\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2N}\right)^2 = \left(\frac{1}{4}+\frac{1}{4N^2}\right)$$

For different values of N, above equation represents a family of circles with centre at x = - ½ , y = 1 / 2 N and radius as

$$\sqrt{\frac{1}{4}+\frac{1}{4N^2}}$$

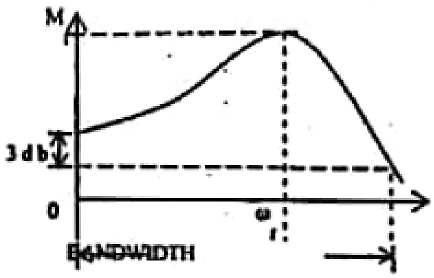
On a particular circle the value of N or the value phase angle of the closed-loop transfer function is constant , therefore, these circles are called N-circles.

ϕ	N = tan ϕ	center x = - ½ , y = 1 / 2N	Radius $R = \sqrt{1/4 + 1/4N^2}$
-90°	∞	0	0.5
-60°	-1.732	-0.289	0.577
-50°	-1.19	-0.42	0.656
-30°	-0.577	0.866	1.0
-10°	-0.176	-2.84	2.88
0°	0	∞	∞
+10°	0.176	2.84	2.88
+30°	0.577	0.866	1.0
+50°	1.19	0.42	0.656
+60°	1.732	0.289	0.577
+90°	∞	0	0.5



Cutoff frequency and Bandwidth :

The closed-loop frequency response of a system is shown in the figure. The response falls by 3 dB from its low frequency value to a frequency value ω_c. The frequency ω_c is called cut
Off frequency and the frequency range 0 to ω_c is called the bandwidth of the system. The resonant
Peak M_r occurs at resonance frequency ω_r.



The bandwidth is defined as the frequency at which The magnitude gain of the frequency response plot reduces to $1/\sqrt{2} = 0.707$; i.e. 3 db of its low frequency value.

For a second order system

$$M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The bandwidth of a second order system having non-zero magnitude at $\omega = 0$ is given by

$$\text{B.W.} = \omega_n \left(1 - 2\zeta^2 + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right)^{1/2}$$

The resonant frequency is $\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$

The resonant magnitude is

$$M_r = \frac{1}{2\zeta\sqrt{(1 - \zeta^2)}}$$

GAIN MARGIN AND PHASE MARGIN

Gain cross - over frequency: The frequency at which the magnitude equal to one or 0 dB
Phase cross over frequency: The frequency at which the phase angle is equal to -180^0.

Gain margin $GM = \frac{1}{|G(j\omega)\,H(j\omega)|}$ **(In Linear)**

The gain margin is a factor by which the gain of a stable system is allowed to increase before the system reaches instability.

The gain margin in dB is

$$G.M = 20 \log_{10} \frac{1}{|G(j\omega_c)\,H(j\omega_c)|} \text{ dB}$$

Procedure to calculate Gain margin :

1. Calculate Phase crossover frequency
   a) by equating phase equation to 180^0 or
   b) by equating imaginary part to zero
2. Calculate the maginitude at phase crossover frequency and is equal to 'a'.
3. Gain margin is equal to 20 log (1 / a).
   For stable systems as $|G(j\omega_c)\,H(j\omega_c)| < 1$, the gain margin in dB is positive.
   For marginally stable systems as $|G(j\omega_c)\,H(j\omega_c)| = 1$, the gain margin in dB is zero.

   For unstable systems as $|G(j\omega_c)\,H(j\omega_c)| > 1$, the gain margin in dB is negative an the gain is to be reduced to make the system is stable.

Phase margin :
The phase margin of a stable system is the amount of additional phase lag required to bring the system to the point of instability.
The phase margin is given by $P.M. = 180^0 + \angle G(s)\,H(s)$

Procedure for calculation of P.M :

1. Calculate 'ω_G' by equating magnitude equation to '1'.
2. Calculate the phase at $\omega = \omega_G$
3. $P.M. = 180^0 + \angle G(s)\,H(s)$.
4. P.M is positive, the system is stable.
   P.M is negative, the system is unstable.
   P.M is zero, the system is marginally stable.